

Split variational inclusion problem and fixed point problem for a class of mappings in $CAT(0)$ spaces.

Yusuf Ibrahim*

Abstract

The aim of this paper is to introduce a modified viscosity iterative method to approximate a solution of split variational inclusion problem and fixed point problem for a uniformly continuous multivalued total quasi-asymptotically strictly pseudocontractive mapping in $CAT(0)$ spaces. Strong convergence theorem for the above problem is established and several important known results are deduced as corollaries to it. As application, a split minimization problem and fixed point problem is solved. A numerical example in support of the main result is given. It seems that the main result on the split variational inclusion problem is new in the setting of $CAT(0)$ spaces.

1 Introduction and Preliminaries

Let X be a metric space. A geodesic path joining $x \in X$ to $y \in X$ is a map $c : [0, l] \subset \mathbb{R} \rightarrow X$ such that $c(0) = x$, $c(l) = y$ and $d(x, y) = l$.

The map c satisfies the isometric property $d(c(t), c(s)) = |t - s|$ for all $t, s \in [0, l]$ and its range is called a geodesic segment joining x and y .

The space (X, d) is said to be a geodesic space if any two points of X are joined by a geodesic segment.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three

*Department of Mathematics, Sa'adatu Rimi College of Education, Kumbotso Kano, P.M.B. 3218 Kano, Nigeria, E-mail: danustazz@gmail.com.

points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ).

A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $(\mathbb{R}^2, d)$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (Bridson and Haefliger [1]).

A metric space X is said to be a $CAT(0)$ space if it is geodesically connected and every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane.

Let Δ be a geodesic triangle in X , and let $\bar{\Delta}$ be its comparison triangle in $\mathbb{R}^2$. Then, X is said to satisfy $CAT(0)$ inequality, if, for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

If $x, y_1, y_2 \in X$, and y_0 is the midpoint of the segment $[y_1, y_2]$, then, the $CAT(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \quad (1.1)$$

It is well known that the following spaces are $CAT(0)$ spaces: a complete, simply connected Riemannian manifold having non-positive sectional curvature; Pre-Hilbert spaces [1]; Euclidean buildings [2]; R-trees [3]; and Hilbert ball with a hyperbolic metric [4, 29].

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and C a nonempty closed and convex subset of H .

The inner product $\langle \cdot, \cdot \rangle : H \times H \rightarrow \mathbb{R}$ generates norm via

$$\langle x, x \rangle = \|x\|^2$$

for all $x \in H$.

A mapping $T : C \rightarrow C$ is said to be total quasi-asymptotically strictly pseudo-contractive (see [30]), if there exists a constant $\gamma \in [0, 1]$ such that

$$\|T^n x - T^n p\|^2 \leq \|x - p\|^2 + \gamma \|x - T^n x\|^2 + \kappa_n \varphi(\|x - p\|) + \mu_n$$

for all $x \in C$ and $p \in F(T)$ where $F(T) = \{x \in X : Tx = x\}$ the fixed point set

of T is nonempty, the sequences $\kappa_n, \mu_n \in [0, \infty)$ satisfy $\lim_{n \rightarrow \infty} \kappa_n = \lim_{n \rightarrow \infty} \mu_n = 0$, and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is strictly increasing and continuous mapping such that $\varphi(0) = 0$.

For concepts such as bounded linear operator and its adjoint operator, maximal monotone operator and metric projection, we refer to Chidume [28].

The metric projection is parity and scale invariant (cf. Proposition 1.26(e) in [14]) in the sense that

$$\lambda P_C x = P_{\lambda C} \lambda x, \text{ for every } \lambda \geq 0, x \in H,$$

consequently,

$$\lambda(P_{C-x}(x) - x) = P_{\lambda C - \lambda x}(\lambda x) - \lambda x, \text{ for every } \lambda \geq 0, x \in H.$$

Fixed point theory in a $CAT(0)$ space has been introduced by Kirk (see for example [5]). He established that a nonexpansive mapping defined on a bounded, closed and convex subset of a complete $CAT(0)$ space has a fixed point. Consequently, fixed point theorems in $CAT(0)$ spaces have been developed by many mathematicians; see for example [6, 7]. More so, some of these theorems in $CAT(0)$ spaces are applicable in many fields of studies such as, graph theory, biology and computer science; see for example [3, 8, 9, 10].

We now set out to establish counterpart of the above concepts in the setting of a $CAT(0)$ space.

Let C be a nonempty subset of a $CAT(0)$ space X .

In [13], a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \rightarrow R$ is said to be quasi-linearization in X if

$$\langle \overrightarrow{pq}, \overrightarrow{rs} \rangle = \frac{1}{2}(d(p, s)^2 + d(q, r)^2 - d(p, r)^2 - d(q, s)^2), \quad (1.2)$$

for all $p, q, r, s \in X$; here a pair $(p, q) \in X \times X$ is denoted by a vector $\overrightarrow{pq}$. Consequently, we have

1. $\langle \overrightarrow{pq}, \overrightarrow{pq} \rangle = d(p, q)^2$,
2. for all $p, q, r, s, t, u, v, w \in X$,

$$\left\langle \overrightarrow{\overrightarrow{pq}r\overrightarrow{s}}, \overrightarrow{\overrightarrow{tuv}\overrightarrow{w}} \right\rangle = \left\langle \overrightarrow{pq}, \overrightarrow{tu} \right\rangle - \left\langle \overrightarrow{pq}, \overrightarrow{vw} \right\rangle - \left\langle \overrightarrow{rs}, \overrightarrow{tu} \right\rangle + \left\langle \overrightarrow{rs}, \overrightarrow{vw} \right\rangle. \quad (1.3)$$

A mapping $T : C \rightarrow C$ is said to be total quasi-asymptotically strictly pseudo-contractive if there exists $\gamma \in [0, 1]$ such that

$$d(T^n x, T^n p)^2 \leq d(x, p)^2 + \gamma d(x, T^n x)^2 + \kappa_n \varphi(d(x, p)) + \mu_n \quad (1.4)$$

for all $x \in C$ and $p \in F(T)$, the sequences $\kappa_n, \mu_n \in [0, \infty)$ satisfy $\lim_{n \rightarrow \infty} \kappa_n = \lim_{n \rightarrow \infty} \mu_n = 0$, and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is strictly increasing and continuous mapping such that $\varphi(0) = 0$.

Denote by $CB(X)$, the collection of all nonempty closed and bounded subsets of X and let H be the Hausdorff metric with respect to the metric d ; that is,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} \quad (1.5)$$

for all $A, B \in CB(X)$, where $d(a, B) = \inf_{b \in B} d(a, b)$ is the distance from the point a to the subset B .

A mapping $A : X \rightarrow X$ is said to be linear if for $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$, we have

$$A(\alpha x \oplus \beta y) = \alpha A(x) \oplus \beta A(y),$$

where $\oplus : X \times X \rightarrow X$ is defined by $(x, y) \mapsto x \oplus y$.

A mapping $A : X \rightarrow X$ is said to be bounded if for all $x, y \in X$, there exists $M \geq 0$ such that

$$d(Ax, Ay)^2 \leq Md(x, y)^2,$$

A mapping $A^* : X \rightarrow X$ is said to be adjoint operator of A if for all $x, y, w, z \in X$, we have

$$\langle \overrightarrow{AxAw}, \overrightarrow{y\dot{z}} \rangle = \langle \overrightarrow{Axw}, \overrightarrow{y\dot{z}} \rangle = \langle \overrightarrow{xw}, \overrightarrow{A^*y\dot{z}} \rangle = \langle \overrightarrow{xw}, \overrightarrow{A^*yA^*z} \rangle. \quad (1.6)$$

Clearly, A^* is a linear operator. As in a Hilbert space, we have

$$d(A^*y, A^*z)^2 = d(Ax, Aw)^2 \leq Md(x, w)^2,$$

and hence, A^* is bounded in X .

A mapping $G : D(G) \subset X \rightarrow 2^X$ is said to be monotone if for any $(x, x^*) \in$

$D(G) \times G(x)$ and $(y, y^*) \in D(G) \times G(y)$, it holds that

$$\langle \overrightarrow{xy}, \overrightarrow{x^*y^*} \rangle \geq 0.$$

A mapping $G : D(G) \subset X \rightarrow 2^X$ is said to be maximal monotone if it is monotone and also has no monotone extension, that is, its graph is not properly contained in the graph of any other monotone operator on X .

For $\gamma > 0$, a mapping $B_\gamma^G = (I + \gamma G)^{-1} : 2^X \rightarrow D(G) \subset X$ is said to be a resolvent of G .

For any $x \in X$, there exists a unique point $x_0 \in C$ such that

$$d(x, x_0) \leq d(x, y) \quad \forall y \in C,$$

and the mapping $P_C : X \rightarrow C$ defined by $P_C x = x_0$ is called the metric projection of X onto C (cf. Proposition 2.4 in [1]), equivalently, in view of the characterization of Hossein and Jamal [19], we have

$$\langle \overrightarrow{x_0x}, \overrightarrow{yx_0} \rangle \geq 0,$$

consequently,

$$P_{\overrightarrow{Cx}} : X \rightarrow \overrightarrow{Cx} \text{ is defined by } \overrightarrow{P_{\overrightarrow{Cx}}(x)x} = \overrightarrow{x_0x},$$

equivalently,

$$\begin{aligned} & \langle \overrightarrow{\overrightarrow{x_0x}x}, \overrightarrow{\overrightarrow{yx_0}x} \rangle \geq 0, \\ & \Leftrightarrow \langle \overrightarrow{x_0x}, \overrightarrow{\overrightarrow{yx_0}x} \rangle \geq 0 \text{ (because } \overrightarrow{x_0x} \text{ is an additive identity element),} \\ & \Leftrightarrow (\langle \overrightarrow{x_0x}, \overrightarrow{yx} \rangle - \langle \overrightarrow{x_0x}, \overrightarrow{x_0x} \rangle) \geq 0 \text{ (by (1.3)),} \end{aligned} \tag{1.7}$$

where $\overrightarrow{x_0x}, \overrightarrow{yx} \in \overrightarrow{Cx}$.

The metric projection is parity and scale invariant in the sense that

$$\lambda P_C x = P_{\lambda C} \lambda x, \text{ for every } \lambda \geq 0, x \in X,$$

consequently,

$$\lambda \overrightarrow{P_{\overrightarrow{Cx}}(x)x} = \overrightarrow{P_{\overrightarrow{\lambda C \lambda x}}(\lambda x) \lambda x}, \text{ for every } \lambda \geq 0, x \in X. \tag{1.8}$$

Let $T : X \rightarrow 2^X$ be a multivalued mapping. A point $x \in X$ is called a fixed point of T if $x \in Tx$ and $F(T) = \{x \in X : x \in Tx\}$ is called the fixed point set of T .

As a generalized version of the well known split common fixed point problem, Moudafi [15] introduced the following split monotone variational inclusion (SMVI) by using maximal monotone mappings;

$$\begin{aligned} \text{find } x^* \in H_1 \text{ such that } 0 \in f(x^*) + B_1(x^*), \\ y^* = Ax^* \in H_2 \text{ solves } 0 \in g(y^*) + B_2(y^*), \end{aligned}$$

where $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$, $A : H_1 \rightarrow H_2$ is a bounded linear operator, $f : H_1 \rightarrow H_1$ and $g : H_2 \rightarrow H_2$ are given single-valued operators.

In 2000, Moudafi [16] proposed the viscosity approximation method by considering the approximate well-posed problem of a nonexpansive mapping S with a contraction mapping f over a nonempty closed and convex subset; in particular he showed that given an arbitrary x_1 in a nonempty closed and convex subset, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) Sx_n,$$

where $\{\alpha_n\} \subset (0, 1)$ with $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, converges strongly to the fixed point set of S , $F(S)$.

In [17], viscosity approximation method for split variational inclusion and fixed point problems in Hilbert Spaces was presented as follows.

$$\begin{cases} u_n &= J_{\lambda}^{B_1}(x_n + \gamma_n A^*(J_{\lambda}^{B_2} - I)Ax_n); \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) T^n(u_n), \forall n \geq 1, \end{cases} \quad (1.9)$$

where B_1 and B_2 are maximal monotone operators, $J_{\lambda}^{B_1}$ and $J_{\lambda}^{B_2}$ are resolvent mappings of B_1 and B_2 respectively, f is a Meir-Keeler mapping, T a nonexpansive mapping, A^* is an adjoint of A , $\gamma_n, \alpha_n \in (0, 1)$ and $\lambda > 0$.

In this paper, motivated by (1.9), a modified viscosity algorithm sequence is presented and prove strong convergence theorem for split variational inclusion problem and fixed point problem of a total quasi-asymptotically strictly pseudocontractive mapping in the setting of two different $CAT(0)$ spaces. It seems that the main result is new in the setting of $CAT(0)$ spaces.

Let X be a complete $CAT(0)$ space and $\{x_n\}$ be a bounded sequence in X with the asymptotic center $A(\{x_n\}) = \{x \in X : \limsup_{n \rightarrow \infty} d(x, x_n) \leq \limsup_{n \rightarrow \infty} d(z, x_n), \forall z \in X\}$. Then $A(\{x_n\})$ consists of exactly one point [20].

Definition 1.1 [11, 12] A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ -convergent to $x \in X$ if x is the unique asymptotic center of any subsequence $\{x_{n_k}\} \subset \{x_n\}$. Symbolically, we write it as $\Delta - \lim_{n \rightarrow \infty} x_n = x$.

Lemma 1.1 [12] Let $\{x_n\}$ be a bounded sequence in a complete $CAT(0)$ space X . Then

- i. $\{x_n\}$ has a Δ -convergent subsequence.
- ii. the asymptotic center of $\{x_n\} \subset C \subset X$ is in C , where C is nonempty, closed and convex.

Lemma 1.2 [25] Let $\{x_n\}$ be a bounded sequence in a complete $CAT(0)$ space and $A(\{x_n\}) = \{x\}$. Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$ and $A(\{x_{n_k}\}) = \{y\}$. If $\lim_{n \rightarrow \infty} d(x_n, y)$ exists, then $x = y$.

Lemma 1.3 [18] Let C be closed and convex subset of a $CAT(0)$ space X and $\{x_n\}$ a bounded sequence in C . Then $\Delta - \lim_{n \rightarrow \infty} x_n = x$ implies that $\{x_n\} \rightharpoonup x$ (i.e. $\limsup_{n \rightarrow \infty} d(x_n, x) = \inf_{y \in C} \limsup_{n \rightarrow \infty} d(x_n, y)$).

Lemma 1.4 [25] Let X be a $CAT(0)$ space and $x, y, z \in X$. Then

- i. $d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z), t \in [0, 1]$,
- ii. $d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2, t \in [0, 1]$.

Lemma 1.5 [21] Let X be a complete $CAT(0)$ space, $\{x_n\}$ a sequence in X and $x \in X$. Then $\{x_n\}$, Δ -converges to x if and only if $\limsup_{n \rightarrow \infty} \langle \overrightarrow{x_n x}, \overrightarrow{y x} \rangle \leq 0$ for all $y \in X$.

Lemma 1.6 [22] Let X be a complete $CAT(0)$ space. Then for all $x, y, z \in X$, the following inequality holds

$$d(x, z)^2 \leq d(y, z)^2 + 2\langle \overrightarrow{xy}, \overrightarrow{xz} \rangle.$$

2 Main results

Let X_1 and X_2 be two $CAT(0)$ spaces, $C \subset X_1$ be a closed and convex subset, $A : X_1 \rightarrow X_2$ bounded linear and unitary operator, $U : X_1 \rightarrow 2^{X_1}$ and $S : X_2 \rightarrow 2^{X_2}$ be uniformly continuous and maximal monotone operators, $f : X_1 \rightarrow X_1$ contraction mapping and $T : C \rightarrow CB(C)$ be uniformly continuous multi-valued total quasi-asymptotically strictly pseudocontractive mapping defined as

$$H(T^n x, T^n p)^2 \leq d(x, p)^2 + \gamma d(x, T^n x)^2 + \kappa_n \varphi(d(x, p)) + \mu_n$$

with the sequences $\kappa_n, \mu_n \in [0, \infty)$ satisfying $\sum_{n=1}^{\infty} \kappa_n < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. Suppose that $\gamma \in [0, 1]$ and $\varphi : [0, \infty) \rightarrow [0, \infty)$ is strictly increasing and continuous mapping such that $\varphi(0) = 0$, and $P_{AC} : X_2 \rightarrow AC$ and $P_{\overrightarrow{ACA\bar{x}}} : X_2 \rightarrow \overrightarrow{ACA\bar{x}}$ are the metric projections onto nonempty closed and convex subsets $AC, \overrightarrow{ACA\bar{x}} \subset X_2$, respectively. Let, for $\gamma > 0$, $B_\gamma^U : 2^{X_1} \rightarrow X_1$ and $B_\gamma^S : 2^{X_2} \rightarrow X_2$ be resolvent operators for U and S , respectively. Denoted by $VIP(U, \gamma)$ and $VIP(S, \gamma)$, and $F(T)$ the solution set of variational inequality problems with respect to U and S and fixed point problem with respect to T . As in [15], we define the split variational inclusion (SVI) as follows:

$$\text{find } x \in X_1 \text{ such that } \overrightarrow{x\bar{x}} \in U(x) \text{ and } Ax \in X_2 \text{ solves } \overrightarrow{Ax\bar{A\bar{x}}} \in S(Ax),$$

where $\overrightarrow{x\bar{x}}$ and $\overrightarrow{Ax\bar{A\bar{x}}}$ are the additive identity elements in X_1 and X_2 , respectively.

Throughout this paper we shall strictly employ the above terminology.

For a bounded sequence $\{x_n\}$ in C , we employ the notion:

$$\limsup_{n \rightarrow \infty} d(x_n, x) = \inf_{y \in C} \limsup_{n \rightarrow \infty} d(x_n, y), \quad (2.1)$$

equivalently x is the asymptotic center of each subsequence of $\{x_n\}$.

First, we establish a demiclosedness principle based on (2.1).

Lemma 2.1 (*Demiclosedness of T*) *Let T be a multi-valued total quasi-asymptotically strictly pseudocontractive mapping on a closed and convex subset C of a $CAT(0)$ space X . Let $\{x_n\}$ be a bounded sequence in C such that $\Delta - \lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then $Tx = x$.*

Proof. By the hypothesis $\Delta - \lim_{n \rightarrow \infty} x_n = x$ and so by Lemma 1.3, we get $\{x_n\} \rightarrow x$. Then by Lemma 1.1 (ii), we arrive at $A(\{x_n\}) = \{x\}$. Let $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. Then we obtain

$$\limsup_{n \rightarrow \infty} d(x_n, y) = \limsup_{n \rightarrow \infty} d(y, Tx_n), \quad (2.2)$$

for all $y \in C$. By (2.2), choosing $y = Tx$, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(x_n, Tx)^2 &= \limsup_{n \rightarrow \infty} H(Tx_n, Tx)^2 \\ &\leq \limsup_{n \rightarrow \infty} \{d(x_n, x)^2 + \gamma d(x_n, Tx_n)^2 + \kappa_n \varphi(d(x_n, x)) + \mu_n\} \\ &\leq \limsup_{n \rightarrow \infty} d(x_n, x)^2. \end{aligned} \quad (2.3)$$

By (1.1), we get

$$d\left(x_n, \frac{x \oplus Tx}{2}\right)^2 \leq \frac{1}{2}d(x_n, x)^2 + \frac{1}{2}d(x_n, Tx)^2 - \frac{1}{4}d(x, Tx)^2.$$

Let $n \rightarrow \infty$ and take superior limit on the both sides of the above inequality and get

$$\begin{aligned} \limsup_{n \rightarrow \infty} d\left(x_n, \frac{x \oplus Tx}{2}\right)^2 &\leq \frac{1}{2} \limsup_{n \rightarrow \infty} d(x_n, x)^2 \\ &\quad + \frac{1}{2} \limsup_{n \rightarrow \infty} d(x_n, Tx)^2 - \frac{1}{4} d(x, Tx)^2. \end{aligned}$$

Since $A(\{x_n\}) = \{x\}$, therefore we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(x_n, x)^2 &\leq \limsup_{n \rightarrow \infty} d\left(x_n, \frac{x \oplus Tx}{2}\right)^2 \leq \frac{1}{2} \limsup_{n \rightarrow \infty} d(x_n, x)^2 \\ &\quad + \frac{1}{2} \limsup_{n \rightarrow \infty} d(x_n, Tx)^2 - \frac{1}{4} d(x, Tx)^2, \end{aligned}$$

which implies that

$$\limsup_{n \rightarrow \infty} d(x_n, x)^2 \leq \limsup_{n \rightarrow \infty} d(x_n, Tx)^2. \quad (2.4)$$

By (2.3) and (2.4), we conclude that $Tx = x$ as desired.

Next, we prove our main result as follows.

Theorem 2.1 *Let $x_1 \in X_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;*

$$\begin{cases} y_n = B_{\gamma_n}^{U_n} \left(\alpha_n x_n \oplus (1 - \alpha_n) \lambda_n A^* \overrightarrow{P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n} \right) \\ x_{n+1} = \beta_n f(x_n) \oplus (1 - \beta_n) z_n, \quad z_n \in \{T_n y_n\}, n \geq 1, \end{cases} \quad (2.5)$$

where A^* is the adjoint operator of A , and $M, \lambda_n, \alpha_n, \beta_n \in [0, 1]$. Suppose that $P_{AC} B_\gamma^S$ is demiclosed and $\Gamma = \{x \in VIP(U, \gamma) : Ax \in VIP(S, \gamma)\} \cap \{x \in F(T)\} \neq \emptyset$, and the following conditions are satisfied;

1. there exists constant $N > 0$ such that $\varphi(r) \leq Nr$, $r \geq 0$;
2. $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \alpha_n = 0$;
3. T satisfies the asymptotically regular condition $\lim_{n \rightarrow \infty} d(y_n, T_n y_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $x \in \Gamma$, where $P_{AC} B_\gamma^S(Ax) = B_\gamma^S(Ax)$.

Proof. We will divide the proof into three steps.

Step one. We prove that $\{x_n\}$ is bounded.

If $p \in \Gamma$, then by Lemma 1.4(ii) and (1.5) we obtain

$$\begin{aligned} d(y_n, p)^2 &= d \left(B_{\gamma_n}^{U_n} \left(\alpha_n x_n \oplus (1 - \alpha_n) \lambda_n A^* \overrightarrow{P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n} \right), p \right)^2 \\ &\leq d \left(\alpha_n x_n \oplus (1 - \alpha_n) \lambda_n A^* \overrightarrow{P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n}, p \right)^2 \\ &\leq \alpha_n d(x_n, p)^2 + (1 - \alpha_n) d \left(\lambda_n A^* \overrightarrow{P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n}, p \right)^2 \end{aligned} \quad (2.6)$$

whereas, by (1.7), (1.3), (1.6) and boundedness, linearity and unitary property

of A , we have

$$\begin{aligned}
& d \left(\lambda_n A^* \overrightarrow{P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n}, p \right)^2 \\
&= \left\langle \overrightarrow{\lambda_n A^* P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n p}, \overrightarrow{\lambda_n A^* P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n p} \right\rangle \\
&= \left\langle \overrightarrow{\lambda_n A^* P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n p}, \overrightarrow{\lambda_n A^* P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n p} \right\rangle \\
&= \left\langle \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n Ap}, \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n Ax_n} \right\rangle \\
&\quad - \left\langle \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n Ap}, \overrightarrow{Ap Ax_n} \right\rangle \\
&= \left\langle \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n Ax_n}, \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n Ax_n} \right\rangle \\
&\quad - \left\langle \overrightarrow{Ap Ax_n}, \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n Ax_n} \right\rangle \\
&\quad - \left\langle \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n Ax_n}, \overrightarrow{Ap Ax_n} \right\rangle + \left\langle \overrightarrow{Ap Ax_n}, \overrightarrow{Ap Ax_n} \right\rangle \\
&\leq - \left\langle \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n Ax_n}, \overrightarrow{Ap Ax_n} \right\rangle + M d(x_n, p)^2. \tag{2.7}
\end{aligned}$$

Substituting (2.7) into (2.6), we get

$$\begin{aligned}
d(y_n, p)^2 &\leq -(1 - \alpha_n) \left\langle \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n Ax_n}, \overrightarrow{Ap Ax_n} \right\rangle \\
&\quad + (M(1 - \alpha_n) + \alpha_n) d(x_n, p)^2 \tag{2.8}
\end{aligned}$$

$$\leq d(x_n, p)^2. \tag{2.9}$$

By (2.5), (1.4), Lemma 1.4(ii) and (1.5), we get

$$\begin{aligned}
d(x_{n+1}, p)^2 &= d(\beta_n f(x_n) \oplus (1 - \beta_n)z_n, p)^2 \\
&\leq \beta_n d(f(x_n), p)^2 + (1 - \beta_n) d(z_n, p)^2 - \beta_n(1 - \beta_n) d(f(y_n), z_n)^2 \\
&\leq \beta_n d(f(x_n), p)^2 + (1 - \beta_n) d(T_n y_n, p)^2 \\
&\leq \beta_n (d(f(x_n), f(p))^2 + d(f(p), p)^2) + (1 - \beta_n) H(T_n y_n, T_n p)^2 \\
&\leq \beta_n \zeta d(x_n, p)^2 + \beta_n d(f(p), p)^2 + (1 - \beta_n) (d(y_n, p)^2 + \gamma d(y_n, T_n y_n)^2 \\
&\quad + \kappa_n \varphi(d(y_n, p)) + \mu_n) \\
&\leq \beta_n d(f(p), p)^2 + (1 - (1 - \zeta)\beta_n)(1 + \kappa_n N) d(x_n, p)^2 \\
&\quad + (1 - \beta_n) \gamma d(y_n, T_n y_n)^2 + (1 - \beta_n) \mu_n.
\end{aligned} \tag{2.10}$$

Since $\sum_{n=1}^{\infty} \kappa_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$ and γ is arbitrary in $[0, 1]$, therefore by (2.10), we get

$$\begin{aligned}
d(x_{n+1}, p)^2 &\leq \beta_n d(f(p), p)^2 + (1 - (1 - \zeta)\beta_n) d(x_n, p)^2 \\
&\leq \max\{d(x_n, p)^2, \frac{1}{1 - \zeta} d(f(p), p)^2\} \\
&\vdots \\
&\leq \max\{d(x_1, p)^2, \frac{1}{1 - \zeta} d(f(p), p)^2\}.
\end{aligned} \tag{2.11}$$

By (2.9) and (2.11), we have that $\{x_n\}$ and $\{y_n\}$ are bounded. Hence $\{T_n y_n\}$ and $\{f(x_n)\}$ are also bounded.

Step two. We will show that $\lim_{n \rightarrow \infty} d(P_{AC_n} B_{\gamma_n}^{S_n}(Ax_n), Ax_n) = 0$.

By Lemmas 1.1 and 1.2, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim_{n \rightarrow \infty} x_{n_j} = x \in C$. Thus, $\Delta - \lim_{n \rightarrow \infty} x_n = x$. By Lemmas 1.5 and 1.6, we get

$$\begin{aligned}
d(x_{n+1}, x_n)^2 &\leq 2d(x_{n+1}, x)^2 + 2d(x, x_n)^2 \\
&= 2\langle \overrightarrow{x_{n+1}x}, \overrightarrow{x_{n+1}x} \rangle + 2\langle \overrightarrow{x_nx}, \overrightarrow{x_nx} \rangle \\
&\longrightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned} \tag{2.12}$$

This implies that $x_n \rightarrow x$ as $n \rightarrow \infty$.

In addition, by Lemma 1.4(ii) we have

$$\begin{aligned}
d(y_n, x_{n+1})^2 &= d(y_n, \beta_n f(x_n) \oplus (1 - \beta_n)z_n)^2 \\
&\leq \beta_n d(y_n, f(x_n))^2 + (1 - \beta_n) d(y_n, z_n)^2 \\
&\leq \beta_n d(y_n, f(x_n))^2 + (1 - \beta_n) d(y_n, T_n y_n)^2 \\
&\longrightarrow 0 \text{ as } n \rightarrow \infty,
\end{aligned} \tag{2.13}$$

and therefore by (2.12), (2.13) and Lemma 1.6, we get

$$\begin{aligned}
d(y_n, x_n)^2 &\leq d(y_n, x_{n+1})^2 + d(x_{n+1}, x_n)^2 \\
&\longrightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned} \tag{2.14}$$

This implies that $y_n \rightarrow x$ as $n \rightarrow \infty$.

As λ_n is arbitrary in $[0, 1]$, so by (2.8), (1.2) and (1.8) we arrive at

$$\begin{aligned}
(1 - \alpha_n) \left\langle \overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n, Ap Ax_n} \right\rangle &\leq d(x_n, p)^2 - d(y_n, p)^2 \\
\implies 2(1 - \alpha_n) d \left(\overrightarrow{\lambda_n P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n, Ax_n} \right)^2 &\leq d(x_n, p)^2 - d(y_n, p)^2 \\
\implies 2(1 - \alpha_n) d \left(\overrightarrow{P_{\lambda_n AC_n \lambda_n Ax_n} B_{\gamma_n}^{S_n}(\lambda_n Ax_n) \lambda_n Ax_n, Ax_n} \right)^2 &\leq d(x_n, p)^2 - d(y_n, p)^2 \\
\implies 2(1 - \alpha_n) d \left(\overrightarrow{P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n, Ax_n} \right)^2 &\leq d(x_n, p)^2 - d(y_n, p)^2 \\
&\longrightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned} \tag{2.15}$$

It follows from (2.15) that

$$d(P_{AC_n} B_{\gamma_n}^{S_n}(Ax_n), Ax_n) \longrightarrow 0 \text{ as } n \rightarrow \infty. \tag{2.16}$$

Step three. We show that $x_n \rightarrow x \in \Gamma$.

By (2.5), we obtain

$$\begin{aligned}
&\alpha_n x_n \oplus (1 - \alpha_n) \lambda_n A^* \overrightarrow{P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n} \in y_n \oplus \gamma_n U_n(y_n) \\
\implies & \\
&x_n \in y_n \oplus \gamma_n U_n(y_n) \text{ (since } \alpha_n \text{ is arbitrary in } [0, 1]). \\
\implies & \\
&\overrightarrow{x_n y_n} \in \gamma_n U_n(y_n)
\end{aligned} \tag{2.17}$$

Since U and S are uniformly continuous, therefore it follows by (2.17), as $n \rightarrow \infty$, that $\overrightarrow{Ax} \in U(x)$. In addition, it is clear that $\Delta - \lim_{n \rightarrow \infty} Ax_n = Ax$. So by using (2.16) and applying the demiclosedness of $P_{AC}B_{\gamma}^S$, we have that $\overrightarrow{Ax} \in SAx$, as $P_{AC}B_{\gamma}^S Ax = B_{\gamma}^S Ax$. On the other hand, by Lemma 2.1 and $\Delta - \lim_{n \rightarrow \infty} y_n = x$ (by (2.14)), we have by the hypothesis $\lim_{n \rightarrow \infty} d(Ty_n, y_n) = 0$ that $x \in Tx$, as T is uniformly continuous. Hence, $x \in \Gamma$.

The proof is completed.

If $U : X_1 \rightarrow X_1$ and $S : X_2 \rightarrow X_2$ are total quasi-asymptotically strictly pseudocontractive in Theorem 2.1 and their fixed point sets $F(U)$ and $F(S)$ are nonempty, then we get:

Corollary 2.1.1 *Let $x_1 \in X_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;*

$$\begin{cases} y_n = U_n \left(\alpha_n x_n \oplus (1 - \alpha_n) \lambda_n A^* \overrightarrow{P_{AC_n Ax_n} S_n(Ax_n) Ax_n} \right) \\ x_{n+1} = \beta_n f(x_n) \oplus (1 - \beta_n) z_n, \quad z_n \in \{T_n y_n\}, n \geq 1, \end{cases}$$

where A^* is the adjoint operator of A , and $M, \lambda_n, \alpha_n, \beta_n \in [0, 1]$. Suppose that $\Gamma = \{x \in F(U) : Ax \in F(S)\} \cap \{x \in F(T)\} \neq \emptyset$, and the following conditions are satisfied;

1. there exists constant $N > 0$ such that $\varphi(r) \leq Nr$, $r \geq 0$;
2. $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \alpha_n = 0$;
3. T satisfies the asymptotically regular condition $\lim_{n \rightarrow \infty} d(y_n, T_n y_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $x \in \Gamma$, where $P_{AC}S(Ax) = S(Ax)$.

Remark 2.1 *Corollary 2.1.1 is about split common fixed point problem and fixed point problem. Hence, this result is new in the literature; in particular, it generalizes similar results in [26, 27] from Banach space setting to CAT(0) spaces.*

In Theorem 2.1, let $\overrightarrow{P_{AC_n Ax_n} S_n(Ax_n) Ax_n} = \overrightarrow{P_{AC_n Ax_n} B_{\gamma_n}^{S_n}(Ax_n) Ax_n}$ and $P_C = B_{\gamma_n}^{U_n}$, where $P_C : X_1 \rightarrow C$ is the metric projection of X_1 onto C . Then we get the following result.

Corollary 2.1.2 *Let $x_1 \in X_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;*

$$\begin{cases} y_n = P_{C_n} \left(\alpha_n x_n \oplus (1 - \alpha_n) \lambda_n A^* \overrightarrow{P_{AC_n A x_n}}(A x_n) A x_n \right) \\ x_{n+1} = \beta_n f(x_n) \oplus (1 - \beta_n) z_n, \quad z_n \in \{T_n y_n\}, n \geq 1, \end{cases}$$

where A^* is the adjoint operator of A , and $M, \lambda_n, \alpha_n, \beta_n \in [0, 1]$. Suppose that $\Gamma = \{x \in C : Ax \in AC\} \cap \{x \in F(T)\} \neq \emptyset$, and the following conditions are satisfied;

1. there exists constant $N > 0$ such that $\varphi(r) \leq Nr$, $r \geq 0$;
2. $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \alpha_n = 0$;
3. T satisfies the asymptotically regular condition $\lim_{n \rightarrow \infty} d(y_n, T_n y_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $x \in \Gamma$.

Remark 2.2 *As Corollary 2.1.2 deals with split feasibility problem and fixed point problem so it is a new result in literature. It also extends similar results in Banach spaces [23, 24] to the case of $CAT(0)$ spaces.*

3 Application to split minimization problem and fixed point problem

The split minimization problem is to find:

$$x \in X_1 \text{ such that } g(x) \leq g(y) \quad \forall y \in X_1$$

and

$$Ax \in X_2 \text{ such that } g'(Ax) \leq g'(Ay) \quad \forall Ay \in X_2$$

where $g : X_1 \rightarrow \mathbb{R}$ and $g' : X_2 \rightarrow \mathbb{R}$ are convex lower semicontinuous functions. Now let the subdifferential of g and g' , $\partial g : X_1 \rightarrow 2^{X_1^*}$ and $\partial g' : X_2 \rightarrow 2^{X_2^*}$ be defined by

$$(\partial g)x = \{x^* \in X_1^* : g(y) - g(x) \geq \langle y - x, x^* \rangle \forall y \in X_1\}$$

and

$$(\partial g')Ax = \{(Ax)^* \in X_2^* : g'(Ay) - g'(Ax) \geq \langle Ay - Ax, (Ax)^* \rangle \forall Ay \in X_2\},$$

respectively.

It is well known that ∂g and $\partial g'$ are maximal monotone on X_1 and X_2 and that $\overrightarrow{xx} \in (\partial g)x$ and $\overrightarrow{AxAx} \in (\partial g')Ax$ iff x and Ax are minimizers of g and g' , respectively. Hence

$$B_\gamma^{\partial g} = \text{prox}_{\gamma g} \text{ and } B_\gamma^{\partial g'} = \text{prox}_{\gamma g'}.$$

In Theorem 2.1, $U = \partial g$ and $S = \partial g'$, give the following result.

Theorem 3.1 *Let the mappings $\partial g, \partial g', \text{prox}_{\gamma g}$ and $\text{prox}_{\gamma g'}$ be defined as above. Let $P_{AC}\text{prox}_{\gamma g'}$ be demiclosed, $x_1 \in X_1$ be chosen arbitrarily and the sequence $\{x_n\}$ be defined as follows;*

$$\begin{cases} y_n = \text{prox}_{\gamma_n g} \left(\alpha_n x_n \oplus (1 - \alpha_n) \lambda_n A^* \overrightarrow{P_{AC_n Ax_n} \text{prox}_{\gamma_n g'}(Ax_n) Ax_n} \right) \\ x_{n+1} = \beta_n f(x_n) \oplus (1 - \beta_n) z_n, \quad z_n \in \{T_n y_n\}, n \geq 1, \end{cases}$$

where A^* is the adjoint operator of A , and $M, \lambda_n, \alpha_n, \beta_n \in [0, 1]$. Suppose that $\Gamma = \{x \in E_1 : g(x) \leq g(y) \text{ and } g'(Ax) \leq g'(Ay) \forall y \in E_1\} \cap \{x \in F(T)\} \neq \emptyset$, and the following conditions are satisfied;

1. there exists constant $N > 0$ such that $\varphi(r) \leq Nr$, $r \geq 0$;
2. $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} \alpha_n = 0$;
3. T satisfies the asymptotically regular condition $\lim_{n \rightarrow \infty} d(y_n, T_n y_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $x \in \Gamma$, where $P_{AC}\text{prox}_{\gamma g'}(Ax) = \text{prox}_{\gamma g'}(Ax)$.

4 A numerical example

Let $X_1 = X_2 = \mathbb{R}$ and $C = AC = [0, \infty)$. Define

$$\begin{aligned}
 U : \mathbb{R} &\rightarrow \mathbb{R} \text{ by } U(x) = \begin{cases} [0, 1), & x \geq 0 \\ \{1\}, & x < 0, \end{cases} \\
 A : \mathbb{R} &\rightarrow \mathbb{R} \text{ by } A(x) = x, x \in \mathbb{R}, \\
 S : \mathbb{R} &\rightarrow \mathbb{R} \text{ by } S(Ax) = \begin{cases} [0, 1), & Ax \geq 0 \\ \{1\}, & Ax < 0, \end{cases} \\
 P_{[0, \infty)} : \mathbb{R} &\rightarrow [0, \infty) \text{ by } P_{[0, \infty)}(Ax) = \begin{cases} 0, & Ax \in (-\infty, 0) \\ Ax, & Ax \in [0, \infty), \end{cases} \\
 (I + \gamma U)^{-1} : \mathbb{R} &\rightarrow \mathbb{R} \text{ by } (I + \gamma U)^{-1}(y) = \begin{cases} \frac{y}{1 + [0, \gamma)}, & y \geq 0 \\ \frac{y}{1 + \gamma}, & y < 0, \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 (I + \gamma S)^{-1} : \mathbb{R} &\rightarrow \mathbb{R} \text{ by } (I + \gamma S)^{-1}(Ay) = \begin{cases} \frac{Ay}{1 + [0, \gamma)}, & Ay \geq 0 \\ \frac{Ay}{1 + \gamma}, & Ay < 0, \end{cases} \\
 P_{[0, \infty)}(I + \gamma S)^{-1} : \mathbb{R} &\rightarrow [0, \infty) \text{ by } P_{[0, \infty)}(I + \gamma S)^{-1}(Ay) = \begin{cases} \frac{Ay}{1 + [0, \gamma)}, & Ay \geq 0 \\ 0, & Ay < 0, \end{cases} \\
 T : \mathbb{R} &\rightarrow \mathbb{R} \text{ by } T(x) = x, \\
 f : \mathbb{R} &\rightarrow \mathbb{R} \text{ by } f(x) = \frac{x}{2}.
 \end{aligned}$$

Clearly, U and S are maximal monotone mappings such that $0 \in VIP(U)$ and $A(0) \in VIP(S)$. Obviously, T is a total quasi-asymptotically strictly pseudo-contractive mapping with sequences $\kappa_n = \mu_n = 0$ and $\gamma = 0$ such that $0 \in F(T)$. Since $\frac{1}{2} \in (0, 1)$, it follows that f is a contraction mapping.

Let us consider $0 \in [0, \gamma)$, $\gamma_n = \log(n + \frac{1}{n})$, $\alpha_n = \frac{n^2 - 2n + 4}{n^3 + 18n - 15}$ and $\beta_n = \frac{e^{\frac{1}{n+1}} - 1}{n}$.

In this case, we have

$$\begin{cases} z_n = \left(\frac{n^2-2n+4}{n^3+18n-15} \right) x_n, x_n \geq 0, \\ z_n = 0, x_n < 0, \\ y_n = z_n, z_n \geq 0, \\ y_n = \frac{z_n}{1+\log(n+\frac{1}{n})}, z_n < 0, \\ x_{n+1} = \frac{e^{\frac{1}{n+1}}-1}{n} \times \frac{x_n}{2} + \left(1 - \frac{e^{\frac{1}{n+1}}-1}{n} \right) y_n, n \geq 1. \end{cases}$$

Now by Theorem 2.1, the sequence $\{x_n\}$ converges strongly to $0 \in \Gamma$. The figures 1 below obtained by (*MATLAB*) software indicates convergence of $\{x_n\}$ given by (2.5) with $x_1 = -15.0$ and $x_1 = 15.0$, respectively. cc

References

- [1] Bridson, M.R. and Haefliger, A.: *Metric Spaces of Non-Positive Curvature*, Vol. 319, Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany (1999).
- [2] Brown, K.S.: *Buildings*, Springer, NewYork, NY, USA (1989).
- [3] Kirk, W. A.: *Fixed point theorems in CAT(0) spaces and R-trees*, Fixed Point Theory Appl., Vol. 2004, No. 4, 309-316 (2004).
- [4] Goebel, K. and Reich, S.: *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, vol. 83 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, NY, USA (1984).
- [5] Kirk, W. A.: *Geodesic geometry and fixed point theory*, in Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), Vol. 64 of Coleccion Abierta, University of Seville, Secretary of Publications, Seville, Spain, 195-225 (2003).
- [6] Sahin, A. and BasarÄsr, M.: *On the strong and Δ -convergence theorems for nonself mappings on a CAT(0) space*, Proceedings of the 10th ICFPTA, July 9-18 2012, Cluj-Napoca, Romania, 227-240 (2012).
- [7] Espinola, R. and Fernandez-Leon, A.: *CAT(k)-spaces, weak convergence and fixed points*, J. Math. Anal. Appl., 353, 410-427 (2009).
- [8] Semple, C. and Steel, M.: *Phylogenetics*, Vol. 24 of Oxford Lecture Series in Mathematics and Its Applications, Oxford University Press, Oxford, UK (2003).

- [9] Kirk, W. A.: *Some recent results in metric fixed point theory*, Journal of Fixed Point Theory Appl., 195-207 (2007).
- [10] Espinola, R. and Kirk, W. A.: *Fixed point theorems in R -trees with applications to graph theory*, Topol. Appl., 153, 1046-1055 (2006).
- [11] Lim, T. C.: *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc., 60, 179-182 (1976).
- [12] Kirk, W. A. and Panyanak, B.: *A concept of convergence in geodesic spaces*, Nonlinear Analysis, 68, 3689-3696 (2008).
- [13] Berg, I. D. and Nikolaev, I. G.: *Quasilinearization and curvature of Alexandrov spaces*, Geom. Dedic. 133, 195-218 (2008).
- [14] Schopfer, F.: *Iterative Methods for the Solution of the Split Feasibility Problem in Banach Spaces*, der Naturwissenschaftlich-Technischen Fakultäten, Universität des Saarlandes (2007).
- [15] Moudafi, A.: *Split monotone variational inclusions*, J. Optim. Theory Appl. 150, 275-283 (2011).
- [16] Moudafi, A.: *Viscosity approximation methods for fixed points problems*, J. Math. Anal. Appl. 241, 46-55 (2000).
- [17] Nimit, N. and Narin, P.: *Viscosity Approximation Methods for Split Variational Inclusion and Fixed Point Problems in Hilbert Spaces*, Proceedings of the International MultiConference of Engineers and Computer Scientists 2014 Vol II, IMECS 2014, March 12 - 14, (2014), Hong Kong.
- [18] Nanjaras, B. and Panyanak, B.: *Demiclosed principle for asymptotically nonexpansive mappings in $CAT(0)$ spaces*, Fixed Point Theory and Applications, 2010, Article ID 268780 (2010).
- [19] Hossein, D. and Jamal, R.: *A characterization of metric projection in $CAT(0)$ Spaces*, Math F.A., 2013.
- [20] Dhompongsa, S., Kirk, W. A. and Sims, B.: *Fixed point of uniformly lipschitzian mappings*, Nonlinear Analysis, 65, 762-772 (2006).
- [21] Kakavandi, B.A.: *Weak topologies in complete $CAT(0)$ Metric spaces*, Proc. Am. Math. Soc. 141, 1029-1039 (2013).
- [22] Wangkeeree, R. and Preechasilp, P.: *Viscosity approximation methods for nonexpansive mappings in $CAT(0)$ spaces*, J. Inequal. Appl. 2013, Article ID 93(2013).
- [23] Khan, A.R., Abbas, M. and Shehu, Y.: *A general convergence theorem for multiple-set split feasibility problem in Hilbert spaces*, Carpathian J. Math., 31, 349-357 (2015).
- [24] Takahashi, W.: *The split feasibility problem in Banach Spaces*, J. Nonlinear Convex Anal. 2014, 15, 1349-1355 (2013).

- [25] Dhompongsa, S. and Panyanak, B.: *On Δ -convergence theorems in $CAT(0)$ spaces*, Computers and Mathematics with Applications, 56, 2572-2579 (2008).
- [26] Tang, J., Chang, S.S., Wang, L. and Wang, X.: *On the split common fixed point problem for strict pseudocontractive and asymptotically nonexpansive mappings in Banach spaces*, J. Inequal. Appl. 2015, 23, 205-221 (2015).
- [27] Moudafi, A.: *A note on the split common fixed point problem for quasi nonexpansive operators*, Nonlinear Anal., 74, 4083-4087 (2011).
- [28] Chidume, C.E.: *Applicable functional analysis*, Ibadan University Press, University of Ibadan, Ibadan, Nigeria (2014).
- [29] Khan, A. R. and Shukri, S. A.: *Best proximity points in the Hilbert ball*, J. Nonlinear Convex Anal., 17, 1083-1094 (2016).
- [30] Chang, S. S. Wang, L. Tang, Y. K. and Yang, L.: *The split common fixed point problem for total asymptotically strictly pseudocontractive mappings*, Journal of Applied Mathematics, 2012, Article ID 385638 (2012).